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ERROR ANALYSIS OF A LINEAR SPLINE
METHOD FOR SOLVING AN
ABEL INTEGRAL EQUATION

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ABSTRACT

A linear spline method for the solution of the Abel integral equation

$$\int_0^x \frac{1}{\sqrt{x^2-s^2}}\, y(s)\,ds = f(x), \qquad x \geq 0$$

is analyzed. The approximate solution along with its derivative converges to the corresponding exact solutions at each point in the interval of integration, the orders of convergence being two and one, respectively. An asymptotic formula for the discretization error is obtained. The method is illustrated by a numerical example.

SIGNIFICANCE AND EXPLANATION

Abel-type integral equations (see Abstract) occur regularly in applications. Typical examples are the determination of the emission coefficients in radiation technology, the determination of gravitational anomalies for an axially symmetric distribution of masses, and the analysis of the fringe shifts in interferograms. Applications of Abel equations are usually directly or indirectly related to systems with axially or radially symmetric geometry.

The equation considered in this paper has an explicit inversion formula. Numerical methods based on this formula have been investigated, but they all have to deal with the presence of a derivative in the inversion formula. Experience shows that direct methods are almost as effective as using the inversion formula, and they can be generalized to solve equations for which an inversion formula is not known. In particular, the results developed in this paper give some idea of the necessary tools and possible results for linear spline or higher degree spline approximate solution for a more general Abel integral equation of the form:

$$\int_0^x \frac{K(x,s)}{(x-s)^\alpha (x+s)^\beta} y(s)\,ds = f(x), \quad 0 < \alpha < 1, \ 0 \le \beta \le 1-\alpha, \ 0 \le x \le 1 \ .$$

The trapezoidal product integration method considered here has been analyzed by Atkinson and Benson. The contribution of the present paper is that we can obtain similar results under slightly weaker assumptions by a much simpler method.

The asymptotic error formula obtained for the numerical solution is simple and quite remarkable because, if we are solving differential or integral equations numerically, we usually have to solve another differential or integral equation to get an error estimate, whereas for Abel equations of the type considered in this paper, the error can be estimated directly from the computed solution.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ERROR ANALYSIS OF A LINEAR SPLINE METHOD

FOR SOLVING AN ABEL INTEGRAL EQUATION

Hing-Sum Hung

1. Introduction.

This paper considers the Abel integral equation:

$$(1.1) \qquad \int_0^x \frac{1}{\sqrt{x^2 - s^2}} \, y(s) \, ds = f(x) \ , \qquad x \geq 0 \ .$$

As described in detail in Section 2, we investigate a direct method for solving this equation based on global approximation of $y(s)$ by a linear spline. Numerical methods based on numerical evaluation of the known explicit inversion formula for (1.1) have been considered in [5], [8]. However, even though the inversion formula is known, it is of interest to investigate direct methods for the above equation, partly because they are almost as effective as using the inversion formula, but mainly because they can be generalized to solve equations for which an inversion formula is not known. Direct methods for the above and related equations have been suggested by a number of authors [1-4, 6, 8, 9, 11, 12].

The trapezoidal product integration method for solving (1.1) considered in the present paper has been analyzed by only two other authors. Atkinson [1] gives a convergence theorem but does not prove that the convergence is $O(h^2)$. Benson [2] obtains $O(h^2)$ and also derives an asymptotic formula for the discretization error, but his method depends on a complicated analysis of product integration. The contribution of the present paper is to obtain similar results under slightly weaker conditions by a much simpler method by using a lemma which is an extension of the lemma stated in [11], p. 179.

Convergence results are given in Section 3. A simple asymptotic formula for the discretization error is derived in Section 4 which confirms the conjecture by Noble [10]. In Section 5, a numerical example is presented.

2. Description of the Method.

Let $x_i = ih$, $i = 0, 1,\ldots,$ where h is an arbitrary constant stepsize. Let Y_i denote an approximation to $y(x_i)$, the exact value of $y(x)$ at $x = x_i$. We use a linear

spline function $P(x)$, with knots at the points x_i, as an approximation to $y(x)$, i.e., for $i = 0, 1, 2, \cdots$

$$(2.1) \qquad P(x) = \frac{1}{h} [(x_{i+1} - x)Y_i + (x - x_i)Y_{i+1}], \quad x_i \leq x \leq x_{i+1} .$$

The function $P(x)$ is continuous at the knots.

The approximate solution of the integral equation is obtained by requiring that (1.1) be satisfied at the knots x_i, i.e., the exact solution $y(x)$ is replaced by the approximate solution $P(x)$ derived from the value $P(x_i) = Y_i$ computed from

$$(2.2) \qquad \int_0^{x_k} \frac{1}{\sqrt{x_k^2 - s^2}} P(s)\, ds = f(x_k), \qquad k = 1, 2, \cdots .$$

This can be rewritten in the form:

$$(2.3) \qquad \sum_{i=0}^{k} w_{k,i}\, Y_i = f(x_k), \qquad k = 1, 2, \cdots ,$$

where

$$(2.4) \qquad \begin{cases} w_{k,0} = \displaystyle\int_{x_0}^{x_1} \frac{1}{\sqrt{x_k^2 - s^2}} \frac{(x_1 - s)}{h}\, ds , \\[2ex] w_{k,i} = \displaystyle\int_{x_{i-1}}^{x_i} \frac{1}{\sqrt{x_k^2 - s^2}} \frac{(s - x_{i-1})}{h}\, ds + \int_{x_i}^{x_{i+1}} \frac{1}{\sqrt{x_k^2 - s^2}} \frac{(x_{i+1} - s)}{h}\, ds, \quad i = 1, \cdots, k-1 , \\[2ex] w_{k,k} = \displaystyle\int_{x_{k-1}}^{x_k} \frac{1}{\sqrt{x_k^2 - s^2}} \frac{(s - x_{k-1})}{h}\, ds . \end{cases}$$

Equation (2.3) is a nonsingular triangular system for Y_i, since

$$(2.5) \qquad \frac{4}{3} \frac{1}{\sqrt{2k}} \leq w_{k,k} \leq \frac{4}{3} \frac{1}{\sqrt{2k-1}}$$

for $k = 1, 2, \cdots$. The starting value of this system needs to be determined by other means, for instance, it might be obtained from

$$(2.6) \qquad Y_0 = y(0) \approx \frac{2}{\pi} f(0) ,$$

which exists, whenever (1.1) has a continuous solution.

An estimate of $y'(x)$ is given by the derivative of (2.1). If we denote this (constant) estimate of $y'(x)$ in $x_i \leq x < x_{i+1}$ by Y'_i, this gives

$$(2.7) \qquad Y'_i = \frac{1}{h} (Y_{i+1} - Y_i), \qquad x_i \leq x < x_{i+1} .$$

3. Convergence of the Method.

The proofs of Theorem 3.1 and Theorem 4.1 require the following lemma.

Lemma 3.1. If there exist a constant $C > 0$ and an integer $N \geq 1$, all independent of k, such that

$$|x_i| \leq C, \qquad\qquad\qquad i = 0, 1, \cdots, N ,$$

$$|x_{k+1}| \leq \sum_{i=0}^{k} |\alpha_{k+1,i}| \ |x_i| + |\beta_k|, \qquad k = N, N+1, \cdots,$$

with

$$\rho_k = 1 - \sum_{i=0}^{k} |\alpha_{k+1,i}| > 0,$$

$$|\beta_k| \leq C\rho_k, \qquad\qquad\qquad k = N, N+1 \cdots,$$

then $\qquad\qquad |x_i| \leq C, \qquad\qquad\qquad i = 0, 1, 2, \cdots .$

Proof: Assume that $|x_i| \leq C$ for $i = 0, 1, \cdots, k$ $(k \geq N)$. Then

$$|x_{k+1}| \leq C \sum_{i=0}^{k} |\alpha_{k+1,i}| + C(1 - \sum_{i=0}^{k} |\alpha_{k+1,i}|) = C .$$

Since this is obviously true for $k = N$, it is by induction true for all $k > N$.

(Note that, if $N = 1$, Lemma 3.1 is identical to the lemma stated in [11, p. 179], which is a consequence of the standard results for regular infinite systems of algebraic equations by Kantorovich and Krylov in [7, p. 27].)

Let $y(x)$ be the exact solution of (1.1), and define the discretization error function by $\epsilon(x) = y(x) - P(x)$, where $P(x)$ is the linear spline function approximation to $y(x)$

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obtained from our numerical method. Denote $\in(x_i)$ by $\in_i$. We state the following theorem:

Theorem 3.1. If $y''(x)$ is Lipschitz continuous on $[0,1]$, then the discretization error of the linear spline method satisfies

$$(3.1) \qquad \in_k = O(h^2), \qquad k = 1, 2, \cdots ,$$

provided $\in_0$, the error of the starting value, is of order h^2.

Proof: By a standard theorem on Lagrange interpolation

$$(3.2) \qquad \in(x) = \frac{1}{h} [(x_{i+1} - x)\in_i + (x - x_i)\in_{i+1}] + \varphi(x) ,$$

where

$$\varphi(x) = \frac{1}{2} y''(\eta_i(x))(x - x_i)(x - x_{i+1}), \qquad x_i \le \eta_i(x) \le x_{i+1}, \; x_i \le x \le x_{i+1} .$$

Since both $y(x)$ and $P(x)$ satisfy (1.1) at each knot $x = x_k$,

$$(3.3) \qquad \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{1}{\sqrt{x_k^2 - s^2}} \in(s) ds = 0, \qquad k = 1, 2, \cdots .$$

This can be rewritten as

$$(3.4) \qquad \sum_{i=0}^{k} w_{k,i} \in_i = R_k, \qquad k = 1, 2, \cdots ,$$

where

$$(3.5) \qquad R_k = - \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{1}{\sqrt{x_k^2 - s^2}} \varphi(s) ds ,$$

and the $w_{k,i}$'s are defined in (2.4).

Multiply (3.4) by k, difference the resulting equation for k and $k + 1$, and then divide by $(k + 1)w_{k+1,k+1}$ to yield the required error equation

$$(3.6) \qquad \in_{k+1} = \sum_{i=0}^{k} a_{k+1,i} \in_i + b_k, \qquad k = 1, 2, \cdots ,$$

where

$$(3.7) \qquad a_{k+1,i} = \frac{k\, w_{k,i} - (k+1)w_{k+1,i}}{(k+1)w_{k+1,k+1}}, \qquad i = 0, 1, \cdots, k ,$$

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and

(3.8) $$b_k = \frac{(k+1)R_{k+1} - k R_k}{(k+1)w_{k+1,k+1}}, \qquad k = 1, 2, \cdots .$$

(Note that (3.6) is not the only possible error equation that can be derived. See, for instance, Atkinson [1] and Benson [2]. However the procedure used here simplifies the asymptotic error analysis in Section 4.)

Equation (3.6) implies that

(3.9) $$|\epsilon_{k+1}| \leq \sum_{i=0}^{k} |a_{k+1,i}| \; |\epsilon_i| + |b_k|, \qquad k = 1, 2, \cdots .$$

Since by assumption, $\epsilon_0 = O(h^2)$, it is easily shown from equation (3.4), by using Lemma 3.3(a) below, that $\epsilon_i = O(h^2)$ for $i = 1, \cdots, K$, with K as defined in Lemma 3.2 below. On the basis of this together with Lemma 3.2(b) and Lemma 3.3(c) below, we can apply Lemma 3.1 on (3.9) and obtain (3.1). Hence the proof of Theorem 3.1 is completed.

Lemma 3.2. If the $a_{k+1,i}$'s are defined by (3.7), then there exists an integer $K \geq 1$, independent of h and k, such that

(a) $a_{k+1,i} \geq 0, \qquad i = 0, 1, \cdots, k-1, \; k \geq 1$,

$\quad\quad a_{k+1,k} \geq 0, \qquad k \geq K$,

(b) $1 - \sum_{i=0}^{k} |a_{k+1,i}| \geq \frac{\pi}{4} \frac{1}{\sqrt{k}}, \qquad k \geq K$.

Proof of (a): From (3.7), using (2.4) and (2.5), we have

$$a_{k+1,0} = \frac{1}{(k+1)w_{k+1,k+1}} \int_{x_0}^{x_1} \left[\frac{1}{\sqrt{1-\left(\frac{s}{x_k}\right)^2}} - \frac{1}{\sqrt{1-\left(\frac{s}{x_{k+1}}\right)^2}} \right] \frac{(x_1-s)}{h^2} \, ds \geq 0 \quad,$$

since the integrand is nonnegative. Similarly, we can prove that $a_{k+1,i} \geq 0$ for $i = 1, \cdots, k-1$. By straightforward estimation, it is easy to show that

$$a_{k+1,k} \geq \frac{1}{w_{k+1,k+1}} \left\{ \frac{4}{3} \frac{1}{\sqrt{2k}} \left[(3 - 2\sqrt{2}) - \frac{1}{k+1} \right] \right\} \quad.$$

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Since the quantity inside the square brackets of the last expression tends to $(3 - 2\sqrt{2}) > 0$ as k increases, there exists an integer K, independent of h and k, such that $a_{k+1,k} \geq 0$ for $k \geq K$.

<u>Proof of (b)</u>: Since by using (2.4) we obtain

$$(3.10) \qquad \sum_{i=0}^{k} w_{k,i} = \int_{0}^{x_k} \frac{1}{\sqrt{x_k^2 - s^2}} \, ds = \frac{\pi}{2}$$

for $k = 1, 2, \cdots$, it can easily be shown that, for $k = 1, 2, \cdots$,

$$(3.11) \qquad \sum_{i=0}^{k} a_{k+1,i} = 1 - \frac{\pi}{2} \frac{1}{(k+1) w_{k+1,k+1}}$$

$$\leq 1 - \frac{\pi}{4} \frac{1}{\sqrt{k}} \quad .$$

By means of part (a) and (3.11), the result of part (b) immediately follows.

<u>Lemma 3.3</u>. If $y''(x)$ is Lipschitz continuous on $[0,1]$, then there exist constants C_1, C_2, $C_3 > 0$, independent of h and k, such that

$$\text{(a)} \quad |F_k| \leq C_1 h^2 \quad ,$$

$$\text{(b)} \quad |R_{k+1} - R_k| \leq C_2 \frac{h^2}{k} \quad ,$$

$$\text{(c)} \quad |b_k| \leq C_3 \frac{h^2}{\sqrt{k}} \quad ,$$

for $k = 1, 2, \cdots$, where R_k and b_k are defined in (3.5) and (3.8), respectively.

<u>Proof of (a)</u>: Let $M_2 = \max\limits_{x \in [0,1]} |y''(x)|$. Then by straightforward estimation, it follows from (3.5) that

$$(3.12) \qquad |R_k| \leq C_1 h^2, \qquad k = 1, 2, \cdots,$$

where $C_1 = \frac{1}{4} M_2$.

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Proof of (b): Subtraction of (3.5) from (3.5) with k replaced by $k + 1$, and by a change of the variable of integration, it is not difficult to show that

$$(3.13) \qquad R_{k+1} - R_k = A_k^{(1)} + A_k^{(2)} + A_k^{(3)}, \qquad k = 1, 2, \cdots ,$$

where

$$A_k^{(1)} = -\frac{1}{2} \sum_{i=0}^{k-1} \int_{x_{i+1}}^{x_{i+2}} \frac{1}{\sqrt{x_{k+1}^2 - s^2}} [y''(\eta_{i+1}(s)) - y''(\eta_i(s-h))] (s-x_{i+1})(s-x_{i+2}) ds ,$$

$$A_k^{(2)} = -\frac{1}{2} \sum_{i=0}^{k-1} \int_{x_{i+1}}^{x_{i+2}} \left[\frac{1}{\sqrt{x_{k+1}+s}} - \frac{1}{\sqrt{x_{k-1}+s}} \right] \frac{1}{\sqrt{x_{k+1}-s}} y''(\eta_i(s-h)) (s-x_{i+1})(s-x_{i+2}) ds ,$$

$$A_k^{(3)} = -\frac{1}{2} \int_{x_0}^{x_1} \frac{1}{\sqrt{x_{k+1}^2 - s^2}} y''(\eta_0(s)) (s-x_0)(s-x_1) ds .$$

Let L_2 be the Lipschitz constant for y''. Then by straightforward estimation and noting that $hk \leq 1$, we obtain from (3.13)

$$(3.14) \qquad |R_{k+1} - R_k| \leq \frac{1}{2} L_2 h^3 + \frac{1}{4} M_2 \frac{h^2}{k} + \frac{1}{8} M_2 \frac{h^2}{k} \leq C_2 \frac{h^2}{k}, \qquad k = 1, 2, \cdots ,$$

where $C_2 = \frac{1}{8} (4L_2 + 3M_2)$.

Proof of (c): From (3.8), using (3.12), (3.14) and (2.5), it can easily be shown that, for $k = 1, 2, \cdots,$

$$|b_k| \leq \frac{|R_{k+1} - R_k| + \frac{1}{k+1} |R_k|}{w_{k+1, k+1}} ,$$

$$\leq C_3 \frac{h^2}{\sqrt{k}} ,$$

where $C_3 = \frac{3}{2} (C_1 + C_2)$.

Corollary 3.1. If the assumptions of Theorem 3.1 are satisfied, then for any fixed $x \in [0,1]$,

$$\epsilon(x) = O(h^2), \qquad \epsilon'(x) = O(h) .$$

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Proof: If we express $\psi(x)$ which is defined in (3.2), in integral form, we obtain

$$\psi(x) = \int_{x_i}^{x} (x-s)y''(s)ds - (x-x_i) \int_{x_i}^{x_{i+1}} \frac{(x_{i+1}-s)}{h} y''(s)ds \quad .$$

By means of (3.1), Corollary 3.1 follows immediately from (3.2) and the equation resulting
from differentiating (3.2).

4. An Asymptotic Formula for the Discretization Error.

In this section we obtain an asymptotic formula for the discretization error of our
numerical solution which confirms the conjecture by Noble [10].

Theorem 4.1. If $y \in C^3[0,1]$, then the discretization error for the linear spline method
satisfies

$$(4.1) \qquad \epsilon_k = \frac{h^2}{12} y''(x_k) + O(h^2/\sqrt{k}), \qquad k = 1, 2, \cdots \quad ,$$

provided ϵ_0, the error of the starting value, is of order h^2.

Proof: Since $y(x) \in C^3[0,1]$, *it is not difficult to show that, for* $x \in [x_i, x_{i+1}]$,

$$(4.2) \quad \epsilon(x) = \frac{1}{h} [(x_{i+1} - x)(\epsilon_i - \frac{h^2}{12} y''(x_i)) + (x-x_i)(\epsilon_{i+1} - \frac{h^2}{12} y''(x_{i+1}))] + \hat{\varphi}(x) \quad ,$$

where

$$\hat{\varphi}(x) = \frac{y''(x_i)}{2} [(x-x_i)(x-x_{i+1}) + \frac{h^2}{6}] + \hat{\rho}(x) \quad ,$$

with

$$\hat{\rho}(x) = \frac{y'''(\eta(x))}{6} (x-x_i)^3 - \frac{1}{12} [2y'''(\eta(x_{i+1})) - y'''(\xi(x_{i+1}))]h^2(x-x_i), \quad \begin{array}{l} x_i \le \eta(x) \le x_{i+1} \\ x_i \le \xi(x) \le x_{i+1} \end{array} \quad .$$

By substituting (4.2) into (3.3) we obtain

$$(4.3) \qquad \sum_{i=0}^{k} w_{k,i}(\epsilon_i - \frac{h^2}{12} y''(x_i)) = \hat{R}_k, \qquad k = 1, 2, \cdots \quad ,$$

where

$$(4.4) \qquad \hat{R}_k = - \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{1}{\sqrt{x_k^2-s^2}} \hat{\varphi}(s)ds \quad .$$

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Comparing (4.3) and (3.4), we note that $\hat{R}_k$ is converging faster than R_k. If we derive an error equation in the same way as in the proof of convergence without modifying (4.3), we expect the parts in the derived error equation which correspond to β_k and ρ_k in Lemma 3.1 would have unbalanced rates of convergence, with the former converging faster than the latter. By applying Lemma 3.1 to such an equation, we will fail to obtain the results we expect. To avoid this, we define $\hat{\epsilon}_i = \sqrt{i+1}(\epsilon_i - \frac{h^2}{12} y''(x_i))$ and rewrite (4.3) as

$$(4.5) \qquad \sum_{i=0}^{k} w_{k,i} \frac{1}{\sqrt{i+1}} \hat{\epsilon}_i = \hat{R}_k, \qquad k = 1, 2, \cdots .$$

Now multiply (4.5) by k, difference the resulting equation for k and $k + 1$, then divide by $(k+1)w_{k+1,k+1}/\sqrt{k+2}$ to yield the required error equation

$$(4.6) \qquad \hat{\epsilon}_{k+1} = \sum_{i=0}^{k} \hat{a}_{k+1,i} \hat{\epsilon}_i + \hat{b}_k, \qquad k = 1, 2, \cdots ,$$

where

$$(4.7) \qquad \hat{a}_{k+1,i} = \frac{\sqrt{k+2}}{\sqrt{i+1}} a_{k+1,i}, \qquad i = 0, 1, \cdots, k ,$$

and

$$(4.8) \qquad \hat{b}_k = \frac{\sqrt{k+2}}{(k+1)w_{k+1,k+1}} [(k+1)\hat{R}_{k+1} - k\hat{R}_k], \qquad k = 1, 2, \cdots ,$$

with $a_{k+1,i}$ as defined in (3.7).

Equation (4.6) implies that

$$(4.9) \qquad |\hat{\epsilon}_{k+1}| \le \sum_{i=0}^{k} |\hat{a}_{k+1,i}| \, |\hat{\epsilon}_i| + |\hat{b}_k|, \qquad k = 1, 2, \cdots .$$

Since $\hat{\epsilon}_0 = \epsilon_0 - \frac{h^2}{12} y''(x_0) = O(h^2)$, it is easily shown from equation (4.5) by using Lemma 4.2(a) below, that $\hat{\epsilon}_i = O(h^2)$ for $i = 1, \cdots, \hat{K}$, with $\hat{K}$ as defined in Lemma 4.1 below. On the basis of this together with Lemma 4.1(b) and Lemma 4.2(c) below, we can apply Lemma 3.1 on (4.9) and conclude that

$$\hat{\epsilon}_k = \sqrt{k+1} \, (\epsilon_k - \frac{h^2}{12} y''(x_k)) = O(h^2), \qquad k = 0, 1, 2, \cdots .$$

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Thus the proof of Theorem 4.1 is completed.

Lemma 4.1. If the $\hat{a}_{k+1,i}$'s are defined by (4.7), then there exist integers K and $\hat{K} \geq 1$, independent of h and k, such that

$$\text{(a)} \quad \hat{a}_{k+1,i} \geq 0, \qquad i = 0, 1, \cdots, k-1, \quad k \geq 1 \ ,$$

$$\hat{a}_{k+1,k} \geq 0, \qquad k \geq K \ ,$$

$$\text{(b)} \quad 1 - \sum_{i=0}^{k} |\hat{a}_{k+1,i}| \geq \frac{\pi}{4} \frac{C}{\sqrt{k}} \ , \qquad k \geq \hat{K} \ ,$$

for an appropriate C, $0 < C < 1$.

Proof of (a): By means of Lemma 3.2(a), part (a) follows immediately from (4.7).

Proof of (b): Since it would be very complex if we estimate $\sum_{i=0}^{k} \hat{a}_{k+1,i}$ directly, we introduce $\tilde{a}_{k+1,i}$:

$$\text{(4.10)} \qquad \tilde{a}_{k+1,i} = a_{k+1,i} + \frac{(1-C)w_{k,i}}{(k+1)w_{k+1,k+1}}, \qquad 0 < C < 1, \quad i = 0, 1, \cdots, k \ .$$

By using Lemma 3.2(a) and noting the nonnegativity of the $w_{k,i}$ from (2.4), we obtain

$$\text{(4.11)} \qquad \tilde{a}_{k+1,i} \geq 0, \qquad i = 0, 1, \cdots, k-1, \quad k \geq 1 \ ,$$

$$\tilde{a}_{k+1,k} \geq 0, \qquad k \geq K \ ,$$

where the K is the same K as defined in Lemma 3.2.

Using (3.10) and (4.11) it can easily be verified that

$$\text{(4.12)} \qquad 1 - \sum_{i=0}^{k} |\tilde{a}_{k+1,i}| \geq \frac{\pi}{4} \frac{C}{\sqrt{k}}, \qquad k \geq K \ .$$

If we can show that for an appropriate C, $0 < C < 1$, $\tilde{a}_{k+1,i} - \hat{a}_{k+1,i} \geq 0$, then the proof is completed. To do this, it is sufficient to show that, for $i = 0, 1, \cdots, k$,

$$D_{k+1,i} = (1-C)w_{k,i} - (\frac{\sqrt{k+2}}{\sqrt{i+1}} - 1) \ [kw_{k,i} - (k+1)w_{k+1,i}] \geq 0 \ .$$

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By using (2.4) and letting $s = ht$, it is easily verified that, for $i = 1, \cdots, k-3$,

$$(4.13) \qquad D_{k+1,i} \geq \int_{i-1}^{i} L_{k,i}(t)(t-i+1)dt + \int_{i}^{i+1} L_{k,i}(t)(i+1-t)dt$$

where

$$L_{k,i}(t) = \frac{(1-C)}{\sqrt{k^2-t^2}} - (\frac{\sqrt{k+2}}{\sqrt{i+1}} - 1) \frac{t^2}{(\sqrt{k^2-t^2})^3} \quad .$$

Since for $i - 1 \leq t \leq i + 1$, $i = 1, \cdots, k-3$,

$$L_{k,i}(t) \geq \frac{[\sqrt{1-C}\, k - \sqrt{C}(i+1)]^2 + [2\sqrt{1-C}\sqrt{C}\, k - \sqrt{k+2}\sqrt{i+1}](i+1)}{[\sqrt{k^2 - (i-1)^2}]^3} \geq 0$$

if C is chosen to be $\frac{1}{2}$, it is obvious from (4.13) that $D_{k+1,i} \geq 0$ for $i = 1, \cdots, k-3$. With this value of C, it is also not difficult to show that as k increases, $D_{k+1,i}$

$(i = 0, k - 2, k - 1, k)$ tends to $\frac{1}{4k}$, $\frac{4}{3}(11 - 12\sqrt{3} + 7\sqrt{2}) \frac{1}{\sqrt{2k}}$, $\frac{4}{3}(3\sqrt{3} - 5\sqrt{2} + 2) \frac{1}{\sqrt{2k}}$, and

$\frac{4}{3}(\sqrt{2} - 1) \frac{1}{\sqrt{2k}}$, respectively, and are therefore greater than zero for sufficiently large k.

Thus, for $i = 0, 1, \cdots, k$, there exists an integer $\check{K} \geq 1$, independent of h and k, such that

$$\tilde{a}_{k+1,i} \geq \hat{a}_{k+1,i}, \qquad k \geq \check{K} \quad .$$

By means of part (a), (4.11) and (4.12), it follows that

$$1 - \sum_{i=0}^{k} |\hat{a}_{k+1,i}| \geq 1 - \sum_{i=0}^{k} |\tilde{a}_{k+1,i}| \geq \frac{\pi}{4} \frac{C}{\sqrt{k}} \quad ,$$

for $C = \frac{1}{2}$, and $k \geq \hat{K}$, with $\hat{K} = \max(K, \check{K})$.

Lemma 4.2. If $y \in C^3[0,1]$, then there exist constants $\hat{c}_1, \hat{c}_2, \hat{c}_3 > 0$, independent of h and k, such that

(a) $|\hat{R}_k| \leq \hat{c}_1 \frac{h^2}{\sqrt{k}}$,

(b) $|\hat{R}_{k+1} - \hat{R}_k| \leq \hat{c}_2 \frac{h^2}{k\sqrt{k}}$,

(c) $|\hat{b}_k| \leq \hat{c}_3 \frac{h^2}{\sqrt{k}}$,

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for $k = 1, 2, \cdots$, where $\hat{R}_k$ and $\hat{b}_k$ are defined in (4.4) and (4.8), respectively.

<u>Proof of (a)</u>: By repeated integration by parts, it is not difficult to show that, for $i = 0, 1, \cdots, k-1$,

(4.14)

$$\frac{1}{2} \int_{x_i}^{x_{i+1}} \frac{1}{\sqrt{x_k^2 - s^2}} y''(x_i) [(s-x_i)(s-x_{i+1}) + \frac{h^2}{6}] ds = \frac{1}{24} \int_{x_i}^{x_{i+1}} \frac{x_k^2 + 2s^2}{(x_k^2 - s^2)^{5/2}} y''(x_i)(s-x_i)^2(s-x_{i+1})^2 ds \quad .$$

Using (4.14) we can rewrite (4.4) as

(4.15) $$\hat{R}_k = \hat{A}_k^{(1)} + \hat{A}_k^{(2)}, \qquad k = 1, 2, \cdots \quad ,$$

where

$$\hat{A}_k^{(1)} = -\frac{1}{24} \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{x_k^2 + 2s^2}{(x_k^2 - s^2)^{5/2}} y''(x_i)(s-x_i)^2(s-x_{i+1})^2 ds \quad ,$$

$$\hat{A}_k^{(2)} = -\sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{1}{\sqrt{x_k^2 - s^2}} \hat{\rho}(s) ds \quad ,$$

with $\hat{\rho}(s)$ as defined in (4.2).

Let $M_2 = \max_{x \in [0,1]} |y''(x)|$ and $M_3 = \max_{x \in [0,1]} |y'''(x)|$. Then by straightforward estimation, and noting that $hk \leq 1$, we obtain from (4.15)

(4.16) $$|\hat{R}_k| \leq M_2 \frac{h^2}{\sqrt{k}} + M_3 h^3 \leq \hat{C}_1 \frac{h^2}{\sqrt{k}}, \qquad k = 1, 2, \cdots \quad ,$$

where $\hat{C}_1 = M_2 + M_3$.

<u>Proof of (b)</u>: Subtraction of (4.15) from (4.15), with k replaced by $k + 1$, then by straightforward estimation, and noting that $hk \leq 1$, it is not difficult to show that for $k = 1, 2, \cdots$

(4.17) $$|\hat{R}_{k+1} - \hat{R}_k| \leq |\hat{A}_{k+1}^{(1)} - \hat{A}_k^{(1)}| + |\hat{A}_{k+1}^{(2)} - \hat{A}_k^{(2)}|$$

$$\leq (7M_2 + M_3) \frac{h^2}{k\sqrt{k}} + 5M_3 \frac{h^3}{\sqrt{k}} \leq \hat{C}_2 \frac{h^2}{k\sqrt{k}} \quad ,$$

where $\hat{C}_2 = 7M_2 + 6M_3$.

<u>Proof of (c)</u>: From (4.8), using (4.16), (4.17) and (2.5) we obtain, for $k = 1, 2, \cdots$,

$$|\hat{b}_k| \leq \frac{\sqrt{k+2}}{(k+1)w_{k+1,k+1}} [(k+1)|\hat{R}_{k+1} - \hat{R}_k| + |\hat{R}_k|]$$

$$\leq \hat{c}_3 \frac{h^2}{\sqrt{k}} \quad ,$$

where $\hat{c}_3 = \frac{3}{2}(\hat{c}_1 + 2\hat{c}_2)$.

5. A Numerical Example.

The linear spline method was applied to the following Abel integral equation:

$$\int_0^x \frac{1}{\sqrt{x^2-s^2}} y(s)ds = \frac{\pi}{2} J_0(x) \quad .$$

The exact solution is $y(x) = \cos(x)$.

In Table 5.1(a) and Table 5.1(b) we list the error $\in(x)$ and $\in'(x)$ at knots and at mid-points between the knots, respectively on $[0,3]$ for different stepsizes h. The error $\in(x)$ and $\in'(x)$ satisfy the predicted h^2 and h dependence, respectively. Note also in Table 5.1(b) that the error $\in'(x)$ at the mid-points are actually $O(h^2)$ although we have not proved that this will be the case.

In Table 5.2 we list the actual error (column 2) for the linear spline method and the theoretical error estimate (column 3) computed from equation (4.1) at knots on $[0,1]$ for $h = 0.01$.

Table 5.1(a)
Error at Knots   (h = 0.1)

x	$\in(x)$			$\in'(x)$		
	h	h/3	h/9	h	h/3	h/9
0.0	.0000E 0	.0000E 0	.0000E 0	.3924E-1	.1309E-1	.4363E-2
0.5	-.7126E-3	-.7963E-4	-.8886E-5	.4345E-1	.1458E-1	.4870E-2
1.0	-.4621E-3	-.5067E-4	-.5570E-5	.2625E-1	.8923E-2	.2992E-2
1.5	-.9744E-4	-.8969E-5	-.9239E-6	.2663E-2	.1084E-2	.3823E-3
2.0	.2927E-3	.3503E-4	.3895E-5	-.2158E-1	-.7021E-2	-.2322E-2
2.5	.6119E-3	.7049E-4	.7724E-5	-.4054E-1	-.1341E-1	-.4457E-2
3.0	.7822E-3	.8873E-4	.9662E-5	-.4958E-1	-.1651E-1	-.5501E-2

Table 5.1(b)
Error at Mid-points   (h = 0.1)

x	$\in(x)$			$\in'(x)$		
	h	h/3	h/9	h	h/3	h/9
0.45	.4062E-3	.4360E-4	.4790E-5	.5117E-4	.1452E-4	.1921E-5
0.95	.2341E-3	.2649E-4	.2997E-5	.2763E-3	.3352E-4	.3855E-5
1.45	.1410E-4	.3312E-5	.4433E-6	.3675E-3	.4287E-4	.4717E-5
1.95	-.2076E-3	-.2058E-4	-.2272E-5	.3662E-3	.4150E-4	.4421E-5
2.45	-.3775E-3	-.3938E-4	-.4472E-5	.2744E-3	.2993E-4	.3097E-5
2.95	-.4544E-3	-.4851E-4	-.5580E-5	.1152E-3	.1104E-4	.1099E-5

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Table 5.2
Actual Error and Theoretical Error Estimate
at Knots with h = 0.01

X	ACTUAL ERROR	THEORETICAL ERROR ESTIMATE
0.1	-.8001E-5	-.8291E-5
0.2	-.7957E-5	-.8167E-5
0.3	-.7790E-5	-.7961E-5
0.4	-.7531E-5	-.7676E-5
0.5	-.7191E-5	-.7313E-5
0.6	-.6778E-5	-.6878E-5
0.7	-.6296E-5	-.6373E-5
0.8	-.5752E-5	-.5806E-5
0.9	-.5151E-5	-.5180E-5
1.0	-.4501E-5	-.4503E-5

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REFERENCES

[1] K. Atkinson, "The numerical solution of an Abel integral equation by a product trapezoidal method", Siam J. Num. Anal., 11 (1974), pp. 97-101.

[2] M. P. Benson, "Errors in numerical quadrature for certain singular integrals and the numerical solution of Abel integral equations", Ph.D. thesis, University of Wisconsin-Madison, 1973.

[3] H. Brunner, "The numerical solution of a class of Abel integral equations by piecewise polynomials", J. Comp. Phys. 12 (1973), pp. 412-416.

[4] H. Brunner, "Global solution of the generalized Abel integral equation by implicit interpolation", Math. Comput. 28 (1974), pp. 61-67.

[5] H. Edels, K. Hearne and A. Young, "Numerical solution of the Abel integral equation", J. Math. and Phys. 41 (1962), pp. 62-75.

[6] P. A. W. Holyhead and S. McKee, "Linear multistep methods for the generalized Abel integral equation", submitted for publication.

[7] L. V. Kantorovich and V. I. Krylov, "Approximate methods of higher analysis", 3rd ed., GITTL, Moscow, 1950; English transl., Interscience, New York; Noordhoff, Groningen, 1958.

[8] P. Linz, "Applications of Abel transformations to the numerical solution of problems in electrostatics and elasticity", MRC Technical Summary Report #826, University of Wisconsin-Madison, 1967.

[9] P. Linz and B. Noble, "A numerical method for treating indentation problems", J. Engrg. Math., 5 (1971), pp. 227-231.

[10] B. Noble, "Rigorous error estimates when solving Abel-type Volterra integral equations of the first kind using spline functions", Lecture notes given at Mathematics Research Center, University of Wisconsin-Madison, 1970.

[11] R. Weiss, "Product integration for the generalized Abel equations", Math. Comp. 26 (1972), pp. 177-190.

[12] R. Weiss and R. S. Anderssen, "A product integration method for a class of singular first kind Volterra equations", Numer. Math., 18 (1972), pp. 442-456.

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A linear spline method for the solution of the Abel integral equation

$$\int_0^x \frac{1}{\sqrt{x^2-s^2}} y(s)\,ds = f(x), \qquad x \geq 0$$

is analyzed. The approximate solution along with its derivative converges to the corresponding exact solutions at each point in the interval of integration, the orders of convergence being two and one, respectively. An asymptotic formula for the discretization error is obtained. The method is illustrated by a numerical example.

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